

THE TAU CONSTANT AND THE DISCRETE LAPLACIAN MATRIX OF A METRIZED GRAPH

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ABSTRACT. We express the tau constant of a metrized graph in terms of the discrete Laplacian matrix and its pseudo inverse.

1. INTRODUCTION

Metrized graphs are finite graphs equipped with a distance function on their edges. For a metrized graph Γ , the tau constant $\tau(\Gamma)$ is an invariant which plays important roles in both harmonic analysis on metrized graphs and arithmetic of curves.

T. Chinburg and R. Rumely [CR] introduced a canonical measure μ_{can} of total mass 1 on a metrized graph Γ . The diagonal values of the Arakelov-Green's function $g_{\mu_{can}}(x, x)$ associated to μ_{can} are constant on Γ . M. Baker and Rumely called this constant “the tau constant” of a metrized graph Γ , and denoted it by $\tau(\Gamma)$. They [BR, Conjecture 14.5] posed a conjecture concerning the existence of a universal lower bound for $\tau(\Gamma)$. We call it Baker and Rumely’s lower bound conjecture.

Baker and Rumely [BR] introduced a measure valued Laplacian operator Δ which extends Laplacian operators studied earlier in [CR] and [Zh1]. This Laplacian operator combines the “discrete” Laplacian on a finite graph and the “continuous” Laplacian $-f''(x)dx$ on \mathbb{R} . In terms of spectral theory, the tau constant $\tau(\Gamma)$ is the trace of the inverse operator of Δ with respect to μ_{can} when Γ has total length 1.

The results in [Zh2], [C1, Chapter 4] and [C4] indicate that the tau constant has important applications in arithmetic of curves such as its connection to the Effective Bogomolov Conjecture over function fields.

In the article [C2], various formulas for $\tau(\Gamma)$ are given, and Baker and Rumely’s lower bound conjecture is verified for a number of large families of graphs. It is shown in the article [C3] that this conjecture holds for metrized graphs with edge connectivity more than 4; and proving it for cubic graphs is sufficient to show that it holds for all graphs.

Verifying the Baker and Rumely’s lower bound conjecture in the remaining cases or showing a counter example to this conjecture, and finding metrized graphs with minimal tau constants are interesting and subtle problems. However, except for some special cases, computing the tau constant for metrized graphs with large number of vertices is not an easy task. In this paper, we will give a formula for the tau constant of Γ in terms of the discrete Laplacian matrix L of Γ and its pseudo inverse L^+ . In particular, this formula leads to rapid computation of $\tau(\Gamma)$ by using computer softwares.

Key words and phrases. Metrized graph, the tau constant, voltage function, resistance function, the discrete Laplacian matrix, pseudo inverse.

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In §2, we briefly introduce metrized graphs, Laplacian operator Δ , the canonical measure μ_{can} and the tau constant $\tau(\Gamma)$. We revise the fact that metrized graphs can be interpreted as electric circuits. At the end of §2, we give several formulas concerning the tau constant. In §3, we introduce the discrete Laplacian matrix L of a metrized graph. We recall some of the properties of L and L^+ . We start §4 with a remarkable relation between the resistance on Γ and the pseudo inverse of the discrete Laplacian on Γ [RB2]. Then we derive a number of new identities by combining this relation with the results from §2 and §3. Finally, we express the canonical measure in terms of L and L^+ , and obtain our main result which is the following theorem:

Theorem 1.1. *Let $L = (l_{pq})_{v \times v}$ be the discrete Laplacian matrix of a metrized graph Γ , and let $L^+ = (l_{pq}^+)_{v \times v}$ be its pseudo inverse. Suppose p_i and q_i are the end points of edge e_i of Γ for each $i = 1, 2, \dots, e$, where e is the number of edges in Γ . Then we have*

$$\tau(\Gamma) = -\frac{1}{12} \sum_{e_i \in E(\Gamma)} l_{p_i q_i} \left(\frac{1}{l_{p_i q_i}} + l_{p_i p_i}^+ - 2l_{p_i q_i}^+ + l_{q_i q_i}^+ \right)^2 + \frac{1}{4} \sum_{q, s \in V(\Gamma)} l_{qs} l_{qq}^+ l_{ss}^+ + \frac{1}{v} \text{trace}(L^+).$$

We prove Theorem 1.1 at the end of §4; and we give two examples for the computations of $\tau(\Gamma)$ and μ_{can} .

Note that there is a 1–1 correspondence between the equivalence classes of finite connected weighted graphs, the metrized graphs, and the resistive electric circuits. If an edge e_i of a metrized graph has length L_i , then we have that the resistance along e_i is L_i in the corresponding resistive electric circuit, and that the weight of e_i is $\frac{1}{L_i}$ in the corresponding weighted graph. The identities we show for metrized graphs in this paper are also valid for electrical networks, and they have equivalent forms on a weighted graph.

The results in this paper are more clarified and organized versions of those given in [C1, Sections 5.1, 5.2, 5.3 and 5.4].

2. THE TAU CONSTANT OF A METRIZED GRAPH

A metrized graph Γ is a finite connected graph such that its edges are equipped with a distinguished parametrization. One can find other definitions of metrized graphs in [Ru], [CR], [BR], [Zh1], and [BF].

A metrized graph can have multiple edges and self-loops. For any given $p \in \Gamma$, the number of directions emanating from p will be called the *valence* of p , and will be denoted by $v(p)$. By definition, there can be only finitely many $p \in \Gamma$ with $v(p) \neq 2$.

For a metrized graph Γ , we will denote its set of vertices by $V(\Gamma)$. We require that $V(\Gamma)$ be finite and non-empty and that $p \in V(\Gamma)$ for each $p \in \Gamma$ if $v(p) \neq 2$. For a given metrized graph Γ , it is possible to enlarge the vertex set $V(\Gamma)$ by considering more additional points of valence 2 as vertices.

For a given graph Γ with vertex set $V(\Gamma)$, the set of edges of Γ is the set of closed line segments with end points in $V(\Gamma)$. We will denote the set of edges of Γ by $E(\Gamma)$. However, we will denote the graph obtained from Γ by deletion of the interior points of an edge $e_i \in E(\Gamma)$ by $\Gamma - e_i$.

We denote $\#(V(\Gamma))$ and $\#(E(\Gamma))$ by v and e , respectively. We denote the length of an edge $e_i \in E(\Gamma)$ by L_i . The total length of Γ , which will be denoted by $\ell(\Gamma)$, is given by $\ell(\Gamma) = \sum_{i=1}^e L_i$.

Let $\text{Zh}(\Gamma)$ be the set of all continuous functions $f : \Gamma \rightarrow \mathbb{C}$ such that for some vertex set $V(\Gamma)$, f is C^2 on $\Gamma \setminus V(\Gamma)$ and $f''(x) \in L^1(\Gamma)$. Baker and Rumely [BR] defined the following measure valued *Laplacian* on a given metrized graph. For a function $f \in \text{Zh}(\Gamma)$,

$$(1) \quad \Delta_x(f(x)) = -f''(x)dx - \sum_{p \in V(\Gamma)} \left[\sum_{\vec{v} \text{ at } p} d_{\vec{v}} f(p) \right] \delta_p(x),$$

See the article [BR] for details and for a description of the largest class of functions for which a measure valued Laplacian can be defined.

In the article [CR], a kernel $j_z(x, y)$ giving a fundamental solution of the Laplacian is defined and studied as a function of $x, y, z \in \Gamma$. For fixed z and y it has the following physical interpretation: When Γ is viewed as a resistive electric circuit with terminals at z and y , with the resistance in each edge given by its length, then $j_z(x, y)$ is the voltage difference between x and z , when unit current enters at y and exits at z (with reference voltage 0 at z).

For any x, y, z in Γ , the voltage function $j_z(x, y)$ on Γ is a symmetric function in x and y , and it satisfies $j_x(x, y) = 0$ and $j_x(y, y) = r(x, y)$, where $r(x, y)$ is the resistance function on Γ . For each vertex set $V(\Gamma)$, $j_z(x, y)$ is continuous on Γ as a function of 3 variables. As the physical interpretation suggests, $j_z(x, y) \geq 0$ for all x, y, z in Γ . For proofs of these facts, see the articles [CR], [BR, sec 1.5 and sec 6], and [Zh1, Appendix]. The voltage function $j_z(x, y)$ and the resistance function $r(x, y)$ on a metrized graph were also studied in the articles [BF], [C2].

For any real-valued, signed Borel measure μ on Γ with $\mu(\Gamma) = 1$ and $|\mu|(\Gamma) < \infty$, define the function $j_\mu(x, y) = \int_\Gamma j_\zeta(x, y) d\mu(\zeta)$. Clearly $j_\mu(x, y)$ is symmetric, and is jointly continuous in x and y . T. Chinburg and Rumely [CR] discovered that there is a unique real-valued, signed Borel measure $\mu = \mu_{\text{can}}$ such that $j_\mu(x, x)$ is constant on Γ . The measure μ_{can} is called the *canonical measure*. Baker and Rumely [BR] called the constant $\frac{1}{2}j_\mu(x, x)$ the *tau constant* of Γ and denoted it by $\tau(\Gamma)$.

Lemma 2.1. [BR, Corollary 14.3] *Let $\{\lambda_1, \lambda_2, \lambda_3, \dots\}$ be the set of eigenvalues of the Laplacian Δ with respect to the canonical measure μ_{can} . Then*

$$\ell(\Gamma) \cdot \tau(\Gamma) = \sum_{n=1}^{\infty} \frac{1}{\lambda_n},$$

In particular, If $\ell(\Gamma) = 1$, then $\tau(\Gamma)$ is the trace of the inverse operator of Δ with respect to μ_{can} .

The following theorem gives an explicit description of the canonical measure μ_{can} :

Theorem 2.2. [CR, Theorem 2.11] *Let Γ be a metrized graph. Suppose that L_i is the length of edge e_i and R_i is the effective resistance between the endpoints of e_i in the graph $\Gamma - e_i$. Then we have*

$$\mu_{\text{can}}(x) = \sum_{p \in V(\Gamma)} \left(1 - \frac{1}{2}v(p)\right) \delta_p(x) + \sum_{e_i \in E(\Gamma)} \frac{dx}{L_i + R_i},$$

where $\delta_p(x)$ is the Dirac measure.

Here is another expression for $\tau(\Gamma)$:

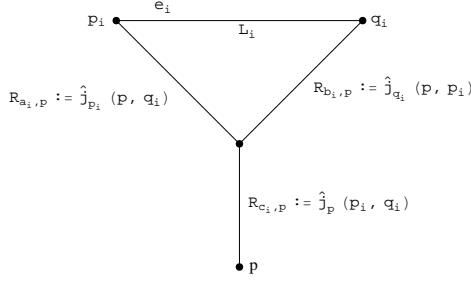


FIGURE 1. Circuit reduction of $\Gamma - e_i$ with reference to p_i , q_i and p .

Lemma 2.3. [REU] *For any metrized graph Γ and its resistance function $r(x, y)$,*

$$\tau(\Gamma) = \frac{1}{2} \int_{\Gamma} r(x, y) d\mu_{can}(y).$$

Another description of $\tau(\Gamma)$ is as follows:

Lemma 2.4. [BR, Lemma 14.4] *For any fixed $p \in \Gamma$, we have $\tau(\Gamma) = \frac{1}{4} \int_{\Gamma} (\frac{d}{dx} r(x, p))^2 dx$.*

Remark 2.5. *Let Γ be any metrized graph with resistance function $r(x, y)$. If we enlarge $V(\Gamma)$ by including points $p \in \Gamma$ with $v(p) = 2$, the resistance function does not change, and thus $\tau(\Gamma)$ does not change by Lemma 2.4.*

Note that $\tau(\Gamma)$ is an invariant of the metrized graph Γ , which depends only on the topology and the edge length distribution of Γ .

Let $\Gamma - e_i$ be a connected graph for an edge $e_i \in E(\Gamma)$ of length L_i . Suppose p_i and q_i are the end points of e_i , and $p \in \Gamma - e_i$. By applying circuit reductions, $\Gamma - e_i$ can be transformed into a Y-shaped graph with the same resistances between p_i , q_i , and p as in $\Gamma - e_i$. More details on this can be found in [C2, Section 2]. Since $\Gamma - e_i$ has such circuit reduction, Γ has the circuit reduction as illustrated in Figure 1 with the corresponding voltage values on each segment, where $\hat{j}_x(y, z)$ is the voltage function in $\Gamma - e_i$. Throughout this paper, we will use the following notation: $R_{a_{i,p}} := \hat{j}_{p_i}(p, q_i)$, $R_{b_{i,p}} := \hat{j}_{q_i}(p_i, p)$, $R_{c_{i,p}} := \hat{j}_p(p_i, q_i)$, and R_i is the resistance between p_i and q_i in $\Gamma - e_i$. Note that $R_{a_{i,p}} + R_{b_{i,p}} = R_i$ for each $p \in \Gamma$. When $\Gamma - e_i$ is not connected, we set $R_{b_{i,p}} = R_i = \infty$ and $R_{a_{i,p}} = 0$ if p belongs to the component of $\Gamma - e_i$ containing p_i , and we set $R_{a_{i,p}} = R_i = \infty$ and $R_{b_{i,p}} = 0$ if p belongs to the component of $\Gamma - e_i$ containing q_i .

By computing the integration in Lemma 2.4, one obtains the following formula for the tau constant:

Proposition 2.6. [REU] *Let Γ be a metrized graph, and let L_i be the length of the edge e_i , for $i \in \{1, 2, \dots, e\}$. Using the notation above, if we fix a vertex p we have*

$$\tau(\Gamma) = \frac{1}{12} \sum_{e_i \in \Gamma} \left(\frac{L_i^3 + 3L_i(R_{a_{i,p}} - R_{b_{i,p}})^2}{(L_i + R_i)^2} \right).$$

Here, if $\Gamma - e_i$ is not connected, i.e. R_i is infinite, the summand corresponding to e_i should be replaced by $3L_i$, its limit as $R_i \rightarrow \infty$.

The proof of Proposition 2.6 can be found in [C2, Proposition 2.9]. We will use the following remark in §4.

Remark 2.7. It follows from Lemma 2.4 and Proposition 2.6 that $\sum_{e_i \in E(\Gamma)} \frac{L_i(R_{a_i,p} - R_{b_i,p})^2}{(L_i + R_i)^2}$ is independent of the chosen vertex $p \in V(\Gamma)$.

Let p_i and q_i be the end points of the edge e_i as in Figure 1. It follows from parallel and series reductions that

$$(2) \quad r(p_i, p) = \frac{(L_i + R_{b_i,p})R_{a_i,p}}{L_i + R_i} + R_{c_i,p}, \quad \text{and} \quad r(q_i, p) = \frac{(L_i + R_{a_i,p})R_{b_i,p}}{L_i + R_i} + R_{c_i,p}.$$

Therefore, $r(p_i, p) - r(q_i, p) = \frac{L_i(R_{a_i,p} - R_{b_i,p})}{L_i + R_i}$, and so

$$(3) \quad \sum_{e_i \in E(\Gamma)} \frac{L_i(R_{a_i,p} - R_{b_i,p})^2}{(L_i + R_i)^2} = \sum_{e_i \in E(\Gamma)} \frac{(r(p_i, p) - r(q_i, p))^2}{L_i}.$$

Proposition 2.8. Let Γ be a metrized graph with the resistance function $r(x, y)$, and let each edge $e_i \in E(\Gamma)$ be parametrized by a segment $[0, L_i]$, under its arclength parametrization. Then for any $p \in V(\Gamma)$,

$$\tau(\Gamma) = -\frac{1}{4} \sum_{q \in V(\Gamma)} (v(q) - 2)r(p, q) + \frac{1}{2} \sum_{e_i \in E(\Gamma)} \frac{1}{L_i + R_i} \int_0^{L_i} r(p, x) dx.$$

Proof. We have $\tau(\Gamma) = \frac{1}{2} \int_{\Gamma} r(p, x) d\mu_{\text{can}}(x)$, by Lemma 2.3. Then by Theorem 2.2,

$$\tau(\Gamma) = \frac{1}{2} \sum_{q \in V(\Gamma)} \left(1 - \frac{1}{2}v(p)\right) \int_{\Gamma} r(p, x) \delta_q(x) + \sum_{e_i \in E(\Gamma)} \frac{1}{L_i + R_i} \int_0^{L_i} r(p, x) dx.$$

This gives the result. \square

Lemma 2.9. Let p_i and q_i be end points of $e_i \in E(\Gamma)$. For any $p \in V(\Gamma)$,

$$\sum_{e_i \in E(\Gamma)} \frac{L_i(R_{a_i,p} - R_{b_i,p})^2}{(L_i + R_i)^2} = \sum_{e_i \in E(\Gamma)} \frac{L_i}{L_i + R_i} (r(p_i, p) + r(q_i, p)) - \sum_{q \in V(\Gamma)} (v(q) - 2)r(p, q).$$

Proof. We first note that $r(x, p) = \frac{(x+R_{a_i,p})(L_i-x+R_{b_i,p})}{L_i+R_i} + R_{c_i,p}$ if $x \in e_i$. By Lemma 2.4, $4\tau(\Gamma) = \int_{\Gamma} \left(\frac{d}{dx}r(x, y)\right)^2 dx$. Thus, integration by parts gives

$$(4) \quad 4\tau(\Gamma) = \sum_{e_i \in E(\Gamma)} \left(r(p, x) \cdot \frac{d}{dx}r(p, x)\right)|_0^{L_i} - \sum_{e_i \in E(\Gamma)} \int_0^{L_i} r(p, x) \frac{d^2}{dx^2}r(p, x) dx.$$

Since $\frac{d^2}{dx^2}r(p, x) = \frac{-2}{L_i+R_i}$ if $x \in e_i$, the result follows from Proposition 2.8 and Equations (2) and (4). \square

Chinburg and Rumely [CR, page 26] showed that

$$(5) \quad \sum_{e_i \in E(\Gamma)} \frac{L_i}{L_i + R_i} = e - v + 1.$$

3. THE DISCRETE LAPLACIAN MATRIX L AND ITS PSEUDO INVERSE L^+ .

Throughout this paper, all matrices will have entries in \mathbb{R} . To have a well-defined discrete Laplacian matrix L for a metrized graph Γ , we first choose a vertex set $V(\Gamma)$ for Γ in such a way that there are no self-loops, and no multiple edges connecting any two vertices. This can be done for any graph Γ by enlarging the vertex set by considering additional valence two points as vertices whenever needed. We will call such a vertex set $V(\Gamma)$ *optimal*. If distinct vertices p and q are the end points of an edge, we call them *adjacent* vertices.

Given a matrix M , let M^T , $tr(M)$, M^{-1} be the transpose, trace and inverse of M , respectively. Let I_v be the $v \times v$ identity matrix, and let O be the zero matrix (with the appropriate size if it is not specified). Let J be an $v \times v$ matrix having each entries 1.

Let Γ be a metrized graph with e edges and with an optimal vertex set $V(\Gamma)$ containing v vertices. Fix an ordering of the vertices in $V(\Gamma)$. Let $\{L_1, L_2, \dots, L_e\}$ be a labeling of the edge lengths. The matrix $A = (a_{pq})_{v \times v}$ given by

$$a_{pq} = \begin{cases} 0 & \text{if } p = q, \text{ or } p \text{ and } q \text{ are not adjacent.} \\ \frac{1}{L_k} & \text{if } p \neq q, \text{ and } p \text{ and } q \text{ are connected by an edge of length } L_k \end{cases}$$

is called the *adjacency matrix* of Γ . Let $D = \text{diag}(d_{pp})$ be the $v \times v$ diagonal matrix given by $d_{pp} = \sum_{s \in V(\Gamma)} a_{ps}$. Then $L := D - A$ is called the *discrete Laplacian matrix* of Γ . That is, $L = (l_{pq})_{v \times v}$ where

$$l_{pq} = \begin{cases} 0 & \text{if } p \neq q, \text{ and } p \text{ and } q \text{ are not adjacent.} \\ -\frac{1}{L_k} & \text{if } p \neq q, \text{ and } p \text{ and } q \text{ are connected by an edge of length } L_k . \\ -\sum_{s \in V(\Gamma) - \{p\}} l_{ps} & \text{if } p = q \end{cases}$$

The discrete Laplacian matrix is also known as the generalized (or the weighted) Laplacian matrix in the literature. A matrix M is called *doubly centered*, if both row and column sums are 0. That is, M is doubly centered iff $MY = O$ and $Y^T M = O$, where $Y = [1, 1, \dots, 1]^T$.

Example 3.1. [C4, Remark 3.1] *For any metrized graph Γ , the discrete Laplacian matrix L is symmetric and doubly centered. That is, $\sum_{p \in V(\Gamma)} l_{pq} = 0$ for each $q \in V(\Gamma)$, and $l_{pq} = l_{qp}$ for each $p, q \in V(\Gamma)$.*

In our case, Γ is connected by definition. Thus, the discrete Laplacian matrix L of Γ is a $(v \times v)$ matrix of rank $v - 1$ if the optimal vertex set $V(\Gamma)$ has v vertices. The null space of L is the 1-dimensional space spanned by $[1, 1, \dots, 1]^T$. Since L is a real symmetric matrix, it has real eigenvalues. Moreover, L is positive semi-definite More precisely, one of the eigenvalues of L is 0 and the others are positive. Thus, L is not invertible. However, it has generalized inverses. In particular, it has the pseudo inverse L^+ , also known as the Moore-Penrose generalized inverse, which is uniquely determined by the following properties:

- | | |
|-----------------------|--------------------------|
| i) $LL^+L = L$, | iii) $(LL^+)^T = LL^+$, |
| ii) $L^+LL^+ = L^+$, | iv) $(L^+L)^T = L^+L$. |

An $v \times v$ matrix M is called an *EP-matrix* if $M^+M = MM^+$. A necessary and sufficient condition for M to be an EP-matrix is that $Mu = \lambda u$ iff $M^+u = \lambda^+ u$, for each eigenvector u of M . Another characterization of an EP-matrix M is that $MX = O$ iff $M^TX = O$, where X is also $v \times v$. Any symmetric matrix is an EP-matrix ([SB, pg 253]).

We have the following properties:

- i) L and L^+ are symmetric,
- iii) L and L^+ are EP matrices,
- ii) L and L^+ are doubly centered,
- iv) L and L^+ are positive semi-definite.

For a discrete Laplacian matrix L of size $v \times v$, we have the following formula for L^+ (see [C-S, ch 10]):

$$(6) \quad L^+ = \left(L - \frac{1}{v}J\right)^{-1} + \frac{1}{v}J.$$

where J is of size $v \times v$ and has all entries 1.

Remark 3.2. Since L^+ is doubly centered, $\sum_{p \in V(\Gamma)} l_{pq}^+ = 0$, for each $q \in V(\Gamma)$. Also, $l_{pq}^+ = l_{qp}^+$, for each $p, q \in V(\Gamma)$.

We use the following lemma and its corollary, Corollary 3.4, frequently in the rest of this article:

Lemma 3.3. [D-M, Equation 2.9] Let J be of size $v \times v$ as above and let L be the discrete Laplacian of a graph (not necessarily with equal edge lengths). Then $LL^+ = L^+L = I - \frac{1}{v}J$.

Corollary 3.4. Let Γ be a metrized graph and let L be the corresponding discrete Laplacian matrix of size $v \times v$. Then for any $p, q \in V(\Gamma)$,

$$\sum_{s \in V(\Gamma)} l_{ps}^+ l_{sq}^+ = \begin{cases} -\frac{1}{v} & \text{if } p \neq q \\ \frac{v-1}{v} & \text{if } p = q \end{cases}.$$

See [C-S], [SB, ch 10], [RB1] and [RM] for more information about L and L^+ .

4. THE DISCRETE LAPLACIAN, THE RESISTANCE FUNCTION, AND THE TAU CONSTANT

In this section, we will obtain a formula (see Theorem 4.10) for the tau constant in terms of the entries of L and L^+ . Our main tools will be a remarkable relation between the resistance and the pseudo inverse L^+ (Lemma 4.1 below), properties of L and L^+ given in §3, the results from §1 concerning metrized graphs, and the circuit reduction theory.

Lemma 4.1. [RB2], [RB3], [D-M, Theorem A] Suppose Γ is a graph with the discrete Laplacian L and the resistance function $r(x, y)$. Let H be a generalized inverse of L (i.e., $LHL = L$). Then we have

$$r(p, q) = H_{pp} - H_{pq} - H_{qp} + H_{qq}, \quad \text{for any } p, q \in V(\Gamma).$$

In particular, for the pseudo inverse L^+ we have

$$r(p, q) = l_{pp}^+ - 2l_{pq}^+ + l_{qq}^+, \quad \text{for any } p, q \in V(\Gamma).$$

Lemma 4.1 shows that the pseudo inverses can be used to compute the resistance $r(p, q)$ between any p, q in Γ . Namely, we choose an optimal vertex set $V(\Gamma)$ containing p and q . Then we compute the corresponding pseudo inverse, and apply Lemma 4.1. Similarly, the following lemma shows that the pseudo inverses can be used to compute the voltage $j_p(q, s)$ for any p, q and s in Γ .

Lemma 4.2. [C4, Lemma 3.5] Let Γ be a graph with the discrete Laplacian L and the voltage function $j_x(y, z)$. Then for any p, q, s in $V(\Gamma)$,

$$j_p(q, s) = l_{pp}^+ - l_{pq}^+ - l_{ps}^+ + l_{qs}^+.$$

Corollary 4.3. Let Γ be a graph with the discrete Laplacian matrix L having the pseudo inverse L^+ . Then for any $p, q \in V(\Gamma)$, we have $l_{pp}^+ \geq l_{pq}^+$.

Proof. By Remark 3.2 and Lemma 4.2, $\sum_{s \in V(\Gamma)} j_p(q, s) = v \cdot (l_{pp}^+ - l_{pq}^+)$ for any p and q in $V(\Gamma)$. Thus the result follows from the fact that $j_p(q, s) \geq 0$ for any $p, q, s \in \Gamma$. \square

Recall that we use L_i for the length of edge $e_i \in E(\Gamma)$ and R_i for the resistance between the endpoints of e_i in the graph $\Gamma - e_i$. An important term for computations concerning $\tau(\Gamma)$ is expressed in terms of L and L^+ by the following lemma:

Lemma 4.4. Let L be the discrete Laplacian matrix of size $v \times v$ for a graph Γ . Let p_i and q_i be the end points of edge e_i for any given $e_i \in E(\Gamma)$. Then

$$\sum_{e_i \in E(\Gamma)} \frac{L_i R_i^2}{(L_i + R_i)^2} = \frac{4(v-1)}{v} \text{tr}(L^+) - \sum_{p, q \in V(\Gamma)} l_{pq} l_{pp}^+ l_{qq}^+ - 2 \sum_{p, q \in V(\Gamma)} l_{pq} (l_{pq}^+)^2.$$

Proof. First, we use Example 3.1 to obtain

$$(7) \quad \sum_{p, q \in V(\Gamma)} l_{pq} (l_{pp}^+)^2 = \sum_{p \in V(\Gamma)} (l_{pp}^+)^2 \left(\sum_{q \in V(\Gamma)} l_{pq} \right) = 0.$$

Using Corollary 3.4,

$$(8) \quad \sum_{p, q \in V(\Gamma)} l_{pq} l_{pq}^+ l_{pp}^+ = \frac{v-1}{v} \cdot \text{tr}(L^+).$$

Then

$$\begin{aligned} \sum_{e_i \in E(\Gamma)} \frac{L_i R_i^2}{(L_i + R_i)^2} &= \sum_{e_i \in E(\Gamma)} \frac{1}{L_i} (r(p_i, q_i))^2, \quad \text{since } r(p_i, q_i) = \frac{L_i R_i}{L_i + R_i}. \\ &= - \sum_{e_i \in E(\Gamma)} l_{p_i q_i} (l_{p_i p_i}^+ + l_{q_i q_i}^+ - 2l_{p_i q_i}^+)^2, \quad \text{by Lemma 4.1}. \\ &= -\frac{1}{2} \sum_{p, q \in V(\Gamma)} l_{pq} (l_{pp}^+ + l_{qq}^+ - 2l_{pq}^+)^2, \quad \text{as } l_{pq} = 0 \text{ if } p, q \text{ are not adjacent}. \\ &= -\frac{1}{2} \sum_{p, q \in V(\Gamma)} l_{pq} (l_{pp}^+ + l_{qq}^+)^2 + 2 \sum_{p, q \in V(\Gamma)} \left(l_{pq} (l_{pp}^+ + l_{qq}^+) l_{pq}^+ - l_{pq} (l_{pq}^+)^2 \right) \\ &= - \sum_{p, q \in V(\Gamma)} l_{pq} l_{pp}^+ l_{qq}^+ + \sum_{p, q \in V(\Gamma)} \left(4l_{pq} l_{pp}^+ l_{pq}^+ - 2l_{pq} (l_{pq}^+)^2 \right), \quad \text{by Equation (7)}. \end{aligned}$$

Thus, the result follows from Equation (8). \square

Next, we will have several lemmas concerning identities involving the entries of L and L^+ .

Lemma 4.5. Let L be the discrete Laplacian matrix of a graph Γ . Then for any $p \in V(\Gamma)$,

$$\sum_{q, s \in V(\Gamma)} l_{qs} (l_{qq}^+ - l_{ss}^+) (l_{qp}^+ - l_{sp}^+) = -2 \sum_{q, s \in V(\Gamma)} l_{qs} l_{qq}^+ l_{sp}^+.$$

Proof. By using Example 3.1, for any $p \in V(\Gamma)$

$$(9) \quad \sum_{q, s \in V(\Gamma)} l_{qs} l_{qq}^+ l_{qp}^+ = \sum_{q \in V(\Gamma)} l_{qq}^+ l_{qp}^+ \left(\sum_{s \in V(\Gamma)} l_{qs} \right) = 0.$$

Using Example 3.1 and Equation (9) for the second equality,

$$\begin{aligned} \sum_{q, s \in V(\Gamma)} l_{qs} (l_{qq}^+ - l_{ss}^+) (l_{qp}^+ - l_{sp}^+) &= \sum_{q, s \in V(\Gamma)} \left(l_{qs} l_{qq}^+ l_{qp}^+ - l_{qs} l_{qq}^+ l_{sp}^+ - l_{qs} l_{ss}^+ l_{qp}^+ + l_{qs} l_{ss}^+ l_{sp}^+ \right) \\ &= - \sum_{q, s \in V(\Gamma)} (l_{qs} l_{qq}^+ l_{sp}^+ + l_{qs} l_{ss}^+ l_{qp}^+). \end{aligned}$$

This is equivalent to what we wanted. \square

Lemma 4.6. *Let L be the discrete Laplacian matrix of size $v \times v$ for a graph Γ , and let p_i, q_i be the end points of $e_i \in E(\Gamma)$. Then for any $p \in V(\Gamma)$,*

$$\begin{aligned} l_{pp}^+ &= \frac{1}{v} \text{tr}(L^+) + \sum_{e_i \in E(\Gamma)} \frac{L_i}{L_i + R_i} (l_{pp_i}^+ + l_{pq_i}^+) - \sum_{q \in V(\Gamma)} v(q) l_{pq}^+, \\ l_{pp}^+ &= \frac{1}{v} \text{tr}(L^+) - \sum_{e_i \in E(\Gamma)} \frac{R_i}{L_i + R_i} (l_{pp_i}^+ + l_{pq_i}^+). \end{aligned}$$

Proof. We use Lemma 2.9 for the first equality below and Lemma 4.1 for the second equality below:

$$\begin{aligned} \sum_{e_i \in E(\Gamma)} \frac{L_i(R_{a_i,p} - R_{b_i,p})^2}{(L_i + R_i)^2} &= \sum_{e_i \in E(\Gamma)} \frac{L_i}{L_i + R_i} (r(p_i, p) + r(q_i, p)) - \sum_{q \in V(\Gamma)} (v(q) - 2)r(p, q) \\ &= \sum_{e_i \in E(\Gamma)} \frac{L_i}{L_i + R_i} (l_{p_ip_i}^+ + l_{q_iq_i}^+ - 2(l_{pp_i}^+ + l_{pq_i}^+ - l_{pp}^+)) - \sum_{q \in V(\Gamma)} (v(q) - 2)(l_{qq}^+ - 2l_{pq}^+ + l_{pp}^+) \\ &= \sum_{e_i \in E(\Gamma)} \frac{L_i}{L_i + R_i} (l_{p_ip_i}^+ + l_{q_iq_i}^+ - 2(l_{pp_i}^+ + l_{pq_i}^+)) + 2l_{pp}^+ - \sum_{q \in V(\Gamma)} (v(q) - 2)l_{qq}^+ + 2 \sum_{q \in V(\Gamma)} v(q)l_{pq}^+. \end{aligned}$$

by Equation (5) and the fact that $\sum_{q \in V(\Gamma)} (v(q) - 2) = 2e - 2v$.

Since $\sum_{e_i \in E(\Gamma)} \frac{L_i(R_{a_i,p} - R_{b_i,p})^2}{(L_i + R_i)^2} = \frac{1}{v} \sum_{p \in V(\Gamma)} \sum_{e_i \in E(\Gamma)} \frac{L_i(R_{a_i,p} - R_{b_i,p})^2}{(L_i + R_i)^2}$ by Remark 2.7, the first equality in the lemma follows if we sum above equality over all $p \in V(\Gamma)$ and apply Example 3.1. Then the second equality in the lemma follows from the fact that $\sum_{q \in V(\Gamma)} v(q)l_{pq}^+ = \sum_{e_i \in E(\Gamma)} (l_{pp_i}^+ + l_{pq_i}^+)$. \square

Lemma 4.7. *Let L be the discrete Laplacian matrix of a graph Γ . Let p_i and q_i be end points of $e_i \in E(\Gamma)$. Then*

$$\sum_{q, s \in V(\Gamma)} l_{qs} l_{qq}^+ l_{ss}^+ = -\frac{1}{2} \sum_{q, s \in V(\Gamma)} l_{qs} (l_{qq}^+ - l_{ss}^+)^2 = \sum_{e_i \in E(\Gamma)} \frac{1}{L_i} (l_{p_ip_i}^+ - l_{q_iq_i}^+)^2 \geq 0.$$

Proof. By Equation (7), $\sum_{q, s \in V(\Gamma)} l_{qs} (l_{qq}^+ - l_{ss}^+)^2 = -2 \sum_{q, s \in V(\Gamma)} l_{qs} l_{qq}^+ l_{ss}^+$. This gives the first equality in the lemma. Then the second equality is obtained by using the definition of L . \square

In Theorem 4.8 below, an important summation term contributing to the tau constant, as can be seen in Proposition 2.6, is expressed in terms of the entries of L and L^+ . This theorem combines various technical lemmas shown above, and it will be used in the proof of Theorem 4.10.

Theorem 4.8. *Let L be the discrete Laplacian matrix of size $v \times v$ for a metrized graph Γ . Let p_i and q_i be end points of edge $e_i \in E(\Gamma)$, and let R_i , $R_{a_i,p}$, $R_{b_i,p}$ and L_i be as defined before.*

$$\sum_{e_i \in E(\Gamma)} \frac{L_i(R_{b_i,p} - R_{a_i,p})^2}{(L_i + R_i)^2} = \frac{4}{v} \text{tr}(L^+) - \frac{1}{2} \sum_{q,s \in V(\Gamma)} l_{qs} (l_{qq}^+ - l_{ss}^+)^2.$$

Proof. Note that the following equality follows from Example 3.1 for any $p \in V(\Gamma)$,

$$(10) \quad \sum_{q,s \in V(\Gamma)} l_{qs} (l_{qp}^+)^2 = \sum_{q \in V(\Gamma)} (l_{qp}^+)^2 \sum_{s \in V(\Gamma)} l_{qs} = 0.$$

By Corollary 3.4, for each $p \in V(\Gamma)$

$$(11) \quad \sum_{q,s \in V(\Gamma)} l_{qs} l_{qp}^+ l_{sp}^+ = l_{pp}^+ - \frac{1}{v} \text{tr}(L^+).$$

Similarly, by Corollary 3.4 and Remark 3.2, for any $p \in V(\Gamma)$ we have

$$(12) \quad \sum_{q,s \in V(\Gamma)} l_{qs} l_{qp}^+ l_{sp}^+ = l_{pp}^+.$$

Then for each $p \in V(\Gamma)$,

$$\begin{aligned} & \sum_{e_i \in E(\Gamma)} \frac{L_i(R_{b_i,p} - R_{a_i,p})^2}{(L_i + R_i)^2} = \sum_{e_i \in E(\Gamma)} \frac{1}{L_i} (r(p_i, p) - r(q_i, p))^2, \quad \text{by Equation (3)} \\ &= - \sum_{e_i \in E(\Gamma)} l_{p_i q_i} (-2l_{pp_i}^+ + l_{p_i p_i}^+ + 2l_{pq_i}^+ - l_{q_i q_i}^+)^2, \quad \text{by Lemma 4.1} \\ &= -\frac{1}{2} \sum_{q,s \in V(\Gamma)} l_{qs} (-2l_{pq}^+ + l_{qq}^+ + 2l_{ps}^+ - l_{ss}^+)^2 \\ &= -\frac{1}{2} \sum_{q,s \in V(\Gamma)} l_{qs} (l_{qq}^+ - l_{ss}^+)^2 + 2 \sum_{q,s \in V(\Gamma)} l_{qs} (l_{qq}^+ - l_{ss}^+) (l_{pq}^+ - l_{ps}^+) - 2 \sum_{q,s \in V(\Gamma)} l_{qs} (l_{pq}^+ - l_{ps}^+)^2 \\ &= -\frac{1}{2} \sum_{q,s \in V(\Gamma)} l_{qs} (l_{qq}^+ - l_{ss}^+)^2 - 4 \sum_{q,s \in V(\Gamma)} l_{qs} l_{qp}^+ l_{sp}^+ - 2 \sum_{q,s \in V(\Gamma)} l_{qs} (l_{pq}^+ - l_{ps}^+)^2, \quad \text{by Lemma 4.5.} \\ &= -\frac{1}{2} \sum_{q,s \in V(\Gamma)} l_{qs} (l_{qq}^+ - l_{ss}^+)^2 - 4 \sum_{q,s \in V(\Gamma)} l_{qs} l_{qp}^+ l_{sp}^+ + 4 \sum_{q,s \in V(\Gamma)} l_{qs} l_{pq}^+ l_{ps}^+, \quad \text{by Equation (10).} \\ &= -\frac{1}{2} \sum_{q,s \in V(\Gamma)} l_{qs} (l_{qq}^+ - l_{ss}^+)^2 - 4(l_{pp}^+ - \frac{1}{v} \text{tr}(L^+)) + 4(l_{pp}^+), \quad \text{by Equations (11) and (12).} \end{aligned}$$

This gives the result. \square

In the following lemma, another important summation term contributing to the tau constant (see Proposition 2.6) is expressed in terms of L and L^+ :

Lemma 4.9. *Let L be the discrete Laplacian matrix of a metrized graph Γ . Suppose p_i and q_i are end points of edge e_i . Then*

$$\sum_{e_i \in E(\Gamma)} \frac{L_i^3}{(L_i + R_i)^2} = \sum_{e_i \in E(\Gamma)} \frac{1}{L_i} (L_i - l_{p_ip_i}^+ + 2l_{p_iq_i}^+ - l_{q_iq_i}^+)^2.$$

Proof. Since $\frac{L_i R_i}{L_i + R_i} = r(p_i, q_i)$ for each $e_i \in E(\Gamma)$, we have

$$\sum_{e_i \in E(\Gamma)} \frac{L_i^3}{(L_i + R_i)^2} = \sum_{e_i \in E(\Gamma)} \frac{1}{L_i} \left(L_i - \frac{L_i R_i}{L_i + R_i} \right)^2 = \sum_{e_i \in E(\Gamma)} \frac{1}{L_i} (L_i - r(p_i, q_i))^2.$$

Then the result follows from Lemma 4.1. \square

Our main result is the following formula for $\tau(\Gamma)$:

Theorem 4.10. *Let L be the discrete Laplacian matrix of size $v \times v$ for a metrized graph Γ , and let L^+ be its pseudo inverse. Suppose p_i and q_i are end points of $e_i \in E(\Gamma)$. Then we have*

$$\begin{aligned} \tau(\Gamma) &= -\frac{1}{12} \sum_{e_i \in E(\Gamma)} l_{p_iq_i} \left(\frac{1}{l_{p_iq_i}} + l_{p_ip_i}^+ - 2l_{p_iq_i}^+ + l_{q_iq_i}^+ \right)^2 + \frac{1}{4} \sum_{q, s \in V(\Gamma)} l_{qs} l_{qq}^+ l_{ss}^+ + \frac{1}{v} \text{tr}(L^+), \\ \tau(\Gamma) &= -\frac{1}{12} \sum_{e_i \in E(\Gamma)} l_{p_iq_i} \left(\frac{1}{l_{p_iq_i}} + l_{p_ip_i}^+ - 2l_{p_iq_i}^+ + l_{q_iq_i}^+ \right)^2 - \frac{1}{4} \sum_{e_i \in E(\Gamma)} l_{p_iq_i} (l_{p_ip_i}^+ - l_{q_iq_i}^+)^2 + \frac{1}{v} \text{tr}(L^+). \end{aligned}$$

Proof. By Proposition 2.6, for any $p \in V(\Gamma)$

$$\tau(\Gamma) = \frac{1}{12} \sum_{e_i \in E(\Gamma)} \frac{L_i^3}{(L_i + R_i)^2} + \frac{1}{4} \sum_{e_i \in E(\Gamma)} \frac{L_i (R_{b_i,p} - R_{a_i,p})^2}{(L_i + R_i)^2}.$$

Thus the first equality in the theorem follows from Lemma 4.9, Theorem 4.8 and Lemma 4.7. Then the second equality follows from Lemma 4.7. \square

Corollary 4.11. *Let L be the discrete Laplacian matrix of size $v \times v$ for a graph Γ . Then we have $\tau(\Gamma) \geq \frac{1}{v} \text{tr}(L^+)$.*

Using Theorem 1.1 and Equation (6), we can compute $\tau(\Gamma)$ and μ_{can} by a computer program whose computational complexity and memory consumption when $\tau(\Gamma)$ is computed is at the level of a matrix inversion.

Next, we will express μ_{can} in terms of the discrete Laplacian matrix and its pseudo inverse.

Proposition 4.12. *For a given metrized graph Γ , let L be its discrete Laplacian, and let L^+ be the corresponding pseudo inverse. Suppose p_i and q_i denotes the end points of $e_i \in E(\Gamma)$. Then we have*

$$\mu_{can}(x) = \sum_{p \in V(\Gamma)} \left(1 - \frac{1}{2} v(p) \right) \delta_p(x) - \sum_{e_i \in E(\Gamma)} \left(l_{p_iq_i} + l_{p_iq_i}^2 (l_{p_ip_i}^+ - 2l_{p_iq_i}^+ + l_{q_iq_i}^+) \right) dx.$$

Proof. The result follows from Theorem 2.2, Lemma 4.1, and the fact that $r(p_i, q_i) = \frac{L_i R_i}{L_i + R_i}$ for each $e_i \in E(\Gamma)$. \square

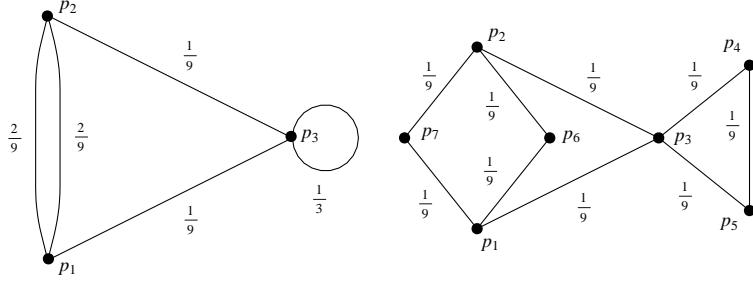


FIGURE 2. Γ with $V(\Gamma) = \{1, 2, 3\}$ and with an optimal vertex set $\{1, 2, 3, 4, 5, 6, 7\}$.

In the rest of this section, we will compute the tau constant and the canonical measure for some metrized graphs.

Example 4.13. Let Γ be a complete graph on 5 vertices with each edge length is equal to $\frac{1}{10}$, so that $\ell(\Gamma) = 1$. Then Γ has the following discrete Laplacian matrix and pseudo inverse:

$$L = \begin{bmatrix} 40 & -10 & -10 & -10 & -10 \\ -10 & 40 & -10 & -10 & -10 \\ -10 & -10 & 40 & -10 & -10 \\ -10 & -10 & -10 & 40 & -10 \\ -10 & -10 & -10 & -10 & 40 \end{bmatrix} \quad \text{and} \quad L^+ = \begin{bmatrix} \frac{2}{125} & -\frac{1}{250} & -\frac{1}{250} & -\frac{1}{250} & -\frac{1}{250} \\ -\frac{1}{250} & \frac{2}{125} & -\frac{1}{250} & -\frac{1}{250} & -\frac{1}{250} \\ -\frac{1}{250} & -\frac{1}{250} & \frac{2}{125} & -\frac{1}{250} & -\frac{1}{250} \\ -\frac{1}{250} & -\frac{1}{250} & -\frac{1}{250} & \frac{2}{125} & -\frac{1}{250} \\ -\frac{1}{250} & -\frac{1}{250} & -\frac{1}{250} & -\frac{1}{250} & \frac{2}{125} \end{bmatrix}.$$

Thus, we obtain $\tau(\Gamma) = \frac{23}{500}$ by applying Theorem 4.10. Moreover, Proposition 4.12 can be used to compute the canonical measure of Γ . Namely,

$$\mu_{can}(x) = - \sum_{p \in V(\Gamma)} \delta_p(x) + 6 \sum_{e_i \in E(\Gamma)} dx.$$

Example 4.14. Let Γ be a metrized graph illustrated as the first graph in Figure 2, where the edge lengths are also shown. Note that $\ell(\Gamma) = 1$. Since Γ with this set of vertices $V(\Gamma) = \{1, 2, 3\}$ has a self loop and two multiple edges, we need to work with an optimal vertex set to have the associated discrete Laplacian matrix. This is done by considering additional 2 points on the self loop as new vertices, and taking 1 more points on each multiple edges as new vertices. The new metrized graph is illustrated by the second graph in Figure 2. As we know by Remark 2.5 that the new length distribution for the self loop and the multiple edges will not change $\tau(\Gamma)$. Now, Γ has the following discrete Laplacian matrix and the pseudo inverse:

$$L = \begin{bmatrix} 27 & 0 & -9 & 0 & 0 & -9 & -9 \\ 0 & 27 & -9 & 0 & 0 & -9 & -9 \\ -9 & -9 & 36 & -9 & -9 & 0 & 0 \\ 0 & 0 & -9 & 18 & -9 & 0 & 0 \\ 0 & 0 & -9 & -9 & 18 & 0 & 0 \\ -9 & -9 & 0 & 0 & 0 & 18 & 0 \\ -9 & -9 & 0 & 0 & 0 & 0 & 18 \end{bmatrix}$$

and

$$L^+ = \begin{bmatrix} \frac{47}{1323} & -\frac{2}{1323} & -\frac{1}{147} & -\frac{10}{441} & -\frac{10}{441} & \frac{4}{441} & \frac{4}{441} \\ -\frac{2}{1323} & \frac{47}{1323} & -\frac{1}{147} & -\frac{10}{441} & -\frac{10}{441} & \frac{4}{441} & \frac{4}{441} \\ -\frac{1}{147} & -\frac{1}{147} & \frac{11}{441} & \frac{4}{441} & \frac{4}{441} & -\frac{13}{882} & -\frac{13}{882} \\ -\frac{10}{441} & -\frac{10}{441} & \frac{4}{441} & \frac{89}{1323} & \frac{40}{1323} & -\frac{3}{98} & -\frac{3}{98} \\ -\frac{10}{441} & -\frac{10}{441} & \frac{4}{441} & \frac{40}{1323} & \frac{89}{1323} & -\frac{3}{98} & -\frac{3}{98} \\ \frac{4}{441} & \frac{4}{441} & -\frac{13}{882} & -\frac{3}{98} & -\frac{3}{98} & \frac{25}{441} & \frac{1}{882} \\ \frac{4}{441} & \frac{4}{441} & -\frac{13}{882} & -\frac{3}{98} & -\frac{3}{98} & \frac{1}{882} & \frac{25}{441} \end{bmatrix}.$$

Finally, applying Theorem 4.10 gives $\tau(\Gamma) = \frac{23}{324}$. By Proposition 4.12, we have the following canonical measure for Γ :

$$\mu_{can}(x) = -\frac{1}{2}\delta_{p_1}(x) - \frac{1}{2}\delta_{p_2}(x) - \delta_{p_3}(x) + 3 \sum_{e_i \in E(\Gamma)} dx.$$

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